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A simple method for the analysis of damped oscillations

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Abstract. We derive and test a simple approximate method for studying damped oscillations. The damping force as a function of velocity can be determined from measurements of the amplitudes of successive oscillations.

1. Introduction

In an experimental investigation by one of the authors (Wraight 1971) the problem arose of analysing damped oscillations where the damping force was an unusual function of velocity. We suggest a simple approximate method which can be used in such cases to obtain the form and magnitude of the unknown force from the successive amplitudes of the damped oscillation. If A is the amplitude of an oscillation of frequency ω, $-\Delta A$ is the decrease in amplitude over half an oscillation, and $f(\dot{x})$ is the damping force when the displacement x is changing at a rate $\dot{x}$, then we show that

$$-\Delta A \simeq \frac{2}{k\omega^2} f\left(\frac{\pi\omega A}{4}\right) \tag{1}$$

where k is the inertia of the system. Thus a plot of $-\Delta A$ against A will give a series of points lying on a curve which, when suitably scaled, will give the damping force as a function of velocity.

If the form of the function f is a power law

$$f = \alpha(\dot{x})^{\nu}$$

for $\dot{x} > 0$ and ν any number greater than 0, not necessarily integral, then a more exact result is

$$-\Delta A \simeq \frac{2}{k\omega^2}\alpha\left(\frac{\pi\omega A}{4}\right)^{\nu}\phi(\nu) \tag{2}$$

where the function $\phi(\nu)$ can be determined.

We thus have a simple and direct method of deducing the damping force, and we present it as one which might be more widely used.

In § 2 we derive equations (1) and (2) using a simple energy-loss argument. A more rigorous discussion is given in the Appendix. In § 3 we test these relations for a selection of functions of the velocity, by comparing them with the solutions of the equation of motion obtained by numerical integration. Some recommendations for the use of the formulae are made in § 4.

2. Derivation of the formulae

We suppose that the displacement x of a system obeys the equation

$$k\ddot{x}+f(\dot{x})+k\omega^2x = 0 \tag{3}$$

$f(\dot{x})$ is assumed to be an odd function of velocity, and to be positive when $\dot{x}$ is positive. We consider the case of light damping, so that the frequency of oscillation may be assumed equal to the undamped frequency ω. We can then take the solution of (3) to be

$$x = A(t)\sin\omega t.$$

Because of the light damping, $A(t)$ will be varying slowly so we can write approximately

$$\dot{x} = \omega A(t)\cos\omega t.$$

The extreme displacements, when $\dot{x} = 0$, then occur at $\sin\omega t = \pm 1$. The energy of the system is $E = \frac{1}{2}k\omega^2A^2$. Consider one half oscillation, from $-A_1$ (minimum) to $+A_2$ (maximum). At each extreme point the kinetic energy is zero, so that the loss of energy is given by

$$\begin{aligned} -\Delta E &= \tfrac{1}{2}k\omega^2(A_1^2-A_2^2) \\ &= -k\omega^2A\,\Delta A \end{aligned} \tag{4}$$

where

$$A = \tfrac{1}{2}(A_1+A_2) \qquad \Delta A = A_2-A_1.$$

But

$$-\Delta E = \int_{-A_1}^{A_2} f(\dot{x})\,\mathrm{d}x. \tag{5}$$

For the case where

$$f(\dot{x}) = \alpha\dot{x}^\nu \qquad (\nu > 0,\text{ not necessarily integral})$$

when $\dot{x} > 0$, we find

$$-\Delta E = \frac{\alpha}{\omega}\int_{-\pi/2}^{\pi/2} (\omega A(t)\cos\omega t)^{\nu+1}\,\mathrm{d}(\omega t).$$

$A(t)$ is varying slowly so we assume it can be replaced by its average value A.

$$\begin{aligned} -\Delta E &= \alpha\omega^\nu A^{\nu+1}\int_{-\pi/2}^{\pi/2} \cos^{\nu+1}\theta\,\mathrm{d}\theta \\ &= 2\alpha\omega^\nu A^{\nu+1}B(\tfrac{1}{2}\nu+1, \tfrac{1}{2}) \end{aligned}$$

where B is the beta function. Then, using equation (4)

$$\begin{aligned} -\Delta A &= \frac{2}{k\omega^2}\alpha(\omega A)^\nu B(\tfrac{1}{2}\nu+1, \tfrac{1}{2}) \\ &= \frac{2}{k\omega^2}\alpha z^\nu\phi(\nu) \end{aligned}$$

where

$$z = \tfrac{1}{4}\pi\omega A$$

and

$$\phi(v) = \frac{1}{2}\left(\frac{4}{\pi}\right)^{v} B\left(\frac{1}{2}v+1, \frac{1}{2}\right).$$

This is the result (2). The function $\phi(v)$ is plotted in figure 1 and it is seen that $\phi(v) \simeq 1$ for $0 < v < 2$. A power law form for the damping will be revealed by a plot of $\ln(-\Delta A)$ against $\ln z$, which will give a straight line of slope v. Once the slope has been measured the value of $\phi(v)$ can be read off from figure 1, and the value of α obtained from the intercept.

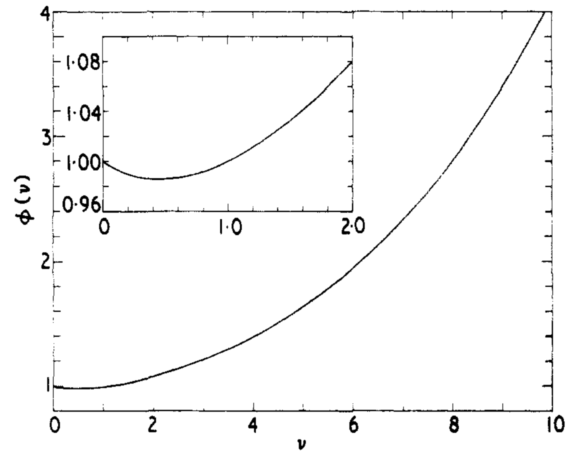


Figure 1. Graph of the function $\phi(v)$ defined in § 2 of the text.

If $f(\dot{x})$ does not have a power law form, we suppose it can be expanded in a power series (for $\dot{x} > 0$)

$$f(\dot{x}) = c_0 + c_1\dot{x} + c_2\dot{x}^2 + \ldots.$$

Then because the energy losses due to each term are additive, we obtain

$$-\Delta A = \frac{2}{k\omega^2}(c_0\phi(0) + c_1 z\phi(1) + c_2 z^2\phi(2) + \ldots). \tag{5}$$

We note that $\phi(0) = \phi(1) = 1$, and $\phi(n)$ does not increase very rapidly with increasing n. Thus if $f(\dot{x})$ is represented well by a few terms of the series, we can write, to a fair degree of approximation

$$-\Delta A \simeq \frac{2}{k\omega^2} f(z)$$

which is the result (1).

3. Tests of the formulae

We tested the formulae for a variety of functions of the velocity by solving the equation

$$\ddot{x} + f(\dot{x}) + x = 0 \tag{6}$$

numerically by means of Runga–Kutta integration, and obtaining values of A and ΔA from the resulting solution for $x(t)$. The program was checked by applying it to cases with a known solution, namely 'frictional' damping (force independent of magnitude of velocity), and 'viscous' damping (force directly proportional to velocity). These tests also showed that the limitation to 'light damping' is not very restrictive. For example, in the case of viscous damping the damping can be so great that the amplitude is reduced by a factor of a half in each oscillation before the error in the formula exceeds 1%. In subsequent calculations we chose the magnitude of the damping to give a reduction in the amplitude by a factor of about ten in twenty oscillations; this was in order to obtain a reasonable number of points on the graphs.

In the case of a power law form for $f(\dot{x})$, there is no detectable deviation from the result (2). For example, figure 2 shows the graph of $\ln(-\Delta A)$ against $\ln z$ for a 7/2 power law.

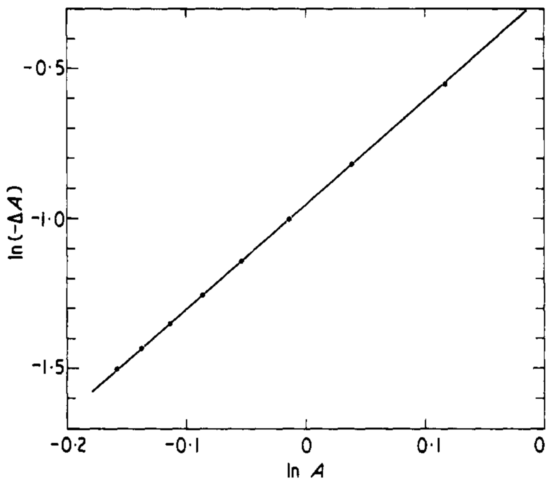


Figure 2. $\mathrm{Ln}(-\Delta A)$ as a function of $\ln A$ for a damping force proportional to the $\frac{7}{2}$ power of the velocity. The points are obtained from values of successive extrema of the numerical solution of the equation of motion. The straight line is the plot obtained from equation (2).

To test the use of equation (1) in situations where the force is not obviously expressible as a power series, we investigated the following three types of damping:

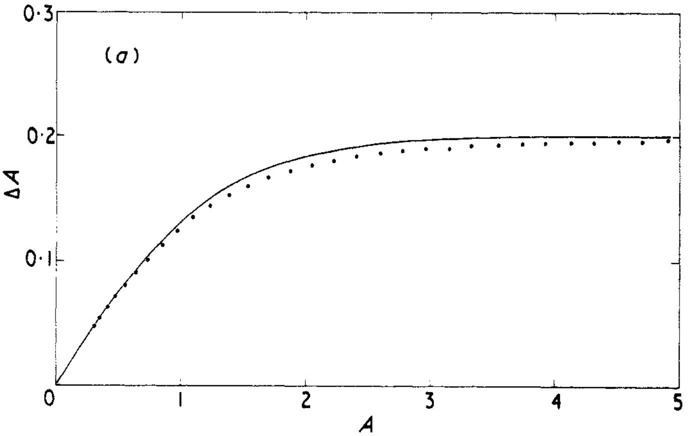
$$f(\dot{x}) = 0{\cdot}1 \tanh(\dot{x}) \tag{7a}$$

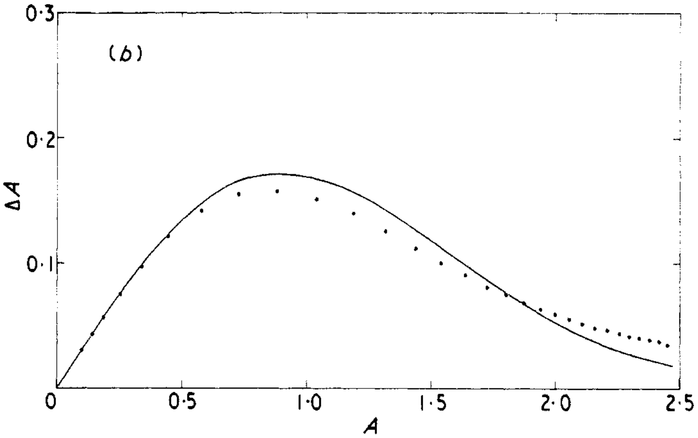
$$f(\dot{x}) = 0{\cdot}2\, \dot{x} \exp(-\dot{x}^2) \tag{7b}$$

and

$$f(\dot{x}) = 0{\cdot}2\left(1 - \frac{\sin 4\dot{x}}{4\dot{x}}\right). \tag{7c}$$

These forms were chosen to give a variety of 'shapes' to the function $f(\dot{x})$. The results are illustrated in figure 3(a, b and c respectively). In each case the plotted points show calculated values of ΔA against A; these are to be compared with the full curves representing $2f(\pi A/4)$. In the first case agreement is excellent, ΔA always being within 5% of $2f(\pi A/4)$. As $A \to 0$, $\tanh A \to A$, and as $A \to \infty$, $\tanh A \to 1$, so in these limits





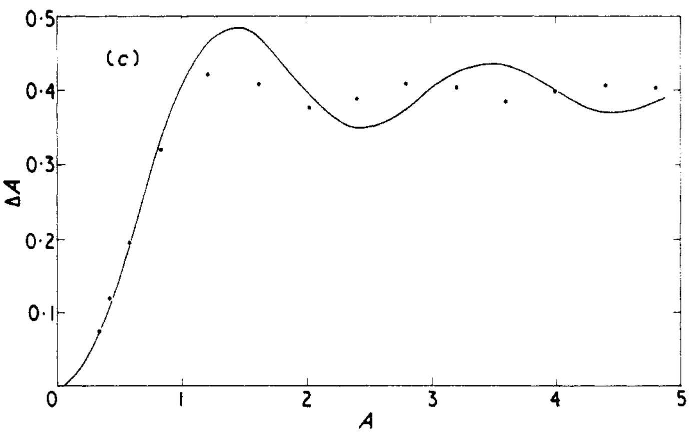


Figure 3. ΔA as a function of A for the damping forces of (*a*) equation (7*a*), (*b*) equation (7*b*) and (*c*) equation (7*c*). In each case the points are obtained from the numerical solution of the equation of motion, whereas the curve is obtained from equation (1).

the agreement is exact. In the second case, where $f(\dot{x})$ increases and then decreases, agreement is still surprisingly good. Even for the 'oscillating' force law of equation (7*c*) the general form of the curve is preserved, though there is a 'phase shift'.

4. Use of the formulae

The first use of equation (1) is to give a general appreciation of the relationship between the shape of damping curves, and the forces which cause them. It becomes obvious that a viscous force proportional to the velocity gives exponential damping; a frictional force independent of the magnitude of the velocity gives linear damping, that is, constant decrease of amplitude per oscillation; and frictional and viscous forces acting together, for example, give exponential damping at high amplitudes, and linear damping at low amplitudes.

Secondly, one can use the formula to find the form of an unusual damping force, as a function of velocity. This increases the usefulness of the study of damped oscillations and might be applied with profit, for example, to the study of damping in liquid helium II (for example, Benson and Hollis-Hallett 1956), as well as superconductors (Wraight 1971). The method has the great advantage of simplicity.

Finally, it should be pointed out that the force must be a function of velocity only, if equation (1) is to be true. A frictional force varying with displacement will give unusual damping, but will not be susceptible to analysis by this method. One can, however, test experimentally whether the force is a function of velocity only, if it is possible to alter the inertia, or the elastic restoring forces, and determine whether the plot of ΔA against A scales according to (1).

Appendix

The result of equation (2) can be derived using the method of asymptotic approximations. Bogoliubov and Mitropolsky (1961) consider the equation

$$k\ddot{x}+k\omega^2 x = \alpha F(\dot{x}).$$

They give as the first approximation to the solution

$$x = A(t)\cos\theta(t)$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\alpha}{2\omega}F_1(A\omega)$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega$$

where

$$F_1(A\omega) = \frac{2}{\pi k}\int_0^\pi F(A\omega\cos\theta)\cos\theta\,\mathrm{d}\theta.$$

The frequency is thus unchanged in this approximation. We are interested in the case where

$$F(\dot{x}) = -\operatorname{sgn}(\dot{x})|\dot{x}|^\nu.$$

We find

$$F_1(A\omega) = -\frac{2}{\pi k}(A\omega)^\nu \int_0^\pi \mathrm{sgn}(\cos\theta)|(\cos\theta)|^\nu \cos\theta \,\mathrm{d}\theta$$

$$= -\frac{4}{\pi k}(A\omega)^\nu \int_0^{\pi/2} \cos^{\nu+1}\theta \,\mathrm{d}\theta$$

giving

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\frac{2\alpha}{\pi k\omega} z^\nu \phi(\nu)$$

in the notation of § 2. Thus the change in amplitude during a half cycle, that is, during an interval of time $t = \pi/\omega$, is

$$-\Delta A = \frac{2\alpha}{k\omega^2} z^\nu \phi(\nu).$$

We can go on to consider the second approximation.

The solution is

$$x = A\cos\theta - \frac{\alpha}{\omega^2}\sum_{\substack{n=0\\ n\neq 1}}^{\infty} \frac{F_n(A\omega)\cos n(\theta+\pi/2)}{n^2-1}$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\alpha}{2\omega}F_1(A\omega)$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega'(A).$$

Here

$$F_n(A\omega) = \frac{2}{\pi k}\int_0^\pi F(A\omega\cos\theta)\cos n\theta \,\mathrm{d}\theta$$

and $\omega'(A)$ is a complicated function of the amplitude. For a force which is always acting against the motion, F_n vanishes for n even. This is because

$$F(A\omega\cos\theta) = \mathrm{sgn}(\cos\theta)F(|A\omega\cos\theta|)$$

is odd about $\theta = \pi/2$ whereas $\cos n\theta$ is even about $\pi/2$ for n even. Thus the extrema, which occur for $\theta = m\pi$, are given by

$$x = \pm A - \frac{\alpha}{\omega^2}\sum_{\substack{n\ \mathrm{odd}\\ n\neq 1}} \frac{F_n(A\omega)\cos n(m\pi+\pi/2)}{n^2-1}$$

$$= \pm A$$

as before.

The difference between successive extrema is then given by

$$-\Delta A = \frac{2\alpha}{\omega\omega'(A)} z^\nu \phi(\nu) \qquad \text{(A.1)}$$

with the only difference from the first approximation being the replacement of π/ω by the actual semiperiod $\pi/\omega'(A)$.

This examination shows that our result equation (2) is exact as long as the frequency of oscillation does not differ significantly from the natural frequency ω. It is well known that large amounts of damping produce only small shifts in frequency. Even if there is a considerable shift of frequency, the damping force can still be obtained by using equation (A.1), that is, by plotting $\ln(-\omega'(A)\,\Delta A)$ as ordinate instead of $\ln(-\Delta A)$.

References

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